EQUATIONS OF THE THEORY OF PLASTICITY WITH ONE-VELOCITY DISLOCATION FLOW

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The well-known difficulties encountered in the theory of plasticity are connected with the fact that, in contrast to elastic distortions β_{ik}, plastic distortions β^P_{ik} cannot be described by simple relations such as Hooke's law [2]. This is due to the dependence of β^P_{ik} on the loading history. Different types of governing relations are adopted in classical strength theories, these relations being generalized empirical equations obtained from tests of macroscopic specimens [2].

A basically different approach is offered by the physical theory of plasticity, in which plastic strain is regarded as the cooperative effect of the motion of dislocations and other defects in the crystalline lattice. Such a description of plastic deformation is statistical and requires the solution of a system of kinetic equations for the distribution functions of the defects [3-5]. In this case, the mathematical problem becomes very complicated, and the advantage of simplicity offered by a phenomenological approach is lost. However, the benefit of physical theory may be not so much in the solution of specific problems as in the proof of various hypotheses which are the basis of phenomenological theories and, in particular, governing relations.

Here attempt to make a link between the physical (dislocation) theory and the phenomenological theory of plasticity. Governing relations are obtained for the rate of plastic distortion β^P_{ik} on the basis of representations of dislocations in the case of their one-velocity flow. We also formulate a closed system of equations for a particular and simple variant of the theory of plasticity. Several specific applications of these equations are given.

1. The elastic behavior of a body under an external load is described by the dynamic equations [2]

$$\sigma^0_{ij,j} + f_i = \rho \ddot{u}^0_i. \tag{1.1}$$

Here, σ^0_{ij} is the tensor of the external elastic stresses; f_i is the density of the body forces; ρ is the density of the body; u^0_i are the elastic displacements; the subscript following the comma denotes differentiation with respect to the corresponding Cartesian coordinate, while the superimposed dot denotes the derivative with respect to time. Closure of system (1.1) requires that the relations for the elastic distortions β^0_{ik} be written in terms of the elastic displacements u^0_i [2]:

$$\beta^0_{ik} = u^0_{k,i}, \tag{1.2}$$

and the use of Hooke's law [2]

$$\sigma^0_{ij} = c_{ijkl}\beta^0_{kl} = c_{ijkl}e^0_{kl}, \tag{1.3}$$

where $c_{ijk\ell}$ are elastic constants of the material; $e^0_{k\ell}$ are elastic strains (the symmetrical part of $\beta^0_{k\ell}$).

The elastoplastic behavior of the body is determined by both the elastic $\beta^0_{k\ell}$ and plastic $\beta^P_{k\ell}$ distortions. Meanwhile, there are no simple relations of the type (1.3) for $\beta^P_{k\ell}$. This is related physically to the complex character of evolution of dislocation ensembles participating in plastic deformation. In the general case, the physical (dislocation) description of plastic deformation entails the solution of the kinetic equation for the distribution functions of the dislocations [3-5]. The complexity of the problem makes it impossible to obtain governing relations directly for the rate of plastic distortion $\dot{\beta}^P_{k\ell}$. However, the

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situation is different in the case of a one-velocity flow of dislocations. Here, it is no longer necessary to solve kinetic equations, and governing relations can be obtained (from first principles) for $\dot{\beta}^P_{k\ell}$.

In the continuum theory of dislocations — which is widely used in the statistical description of dislocations — the main characteristics are the tensors of dislocation density $\alpha_{p\ell}$ and dislocation flux density $J_{k\ell}$. These tensors are connected to the tensor of plastic distortion β^P_{ik} by the formulas [3]

$$\alpha_{pl} = -\varepsilon_{pmk}\beta^P_{kl,m}; \tag{1.4}$$

$$J_{kl} = \dot{\beta}^P_{kl} \tag{1.5}$$

(ε_{pmk} is an antisymmetric unit tensor). On the other hand, the tensors $\alpha_{p\ell}$ and $J_{k\ell}$ can be expressed through the dislocation distribution function $f(\tau, b; r, t)$ by means of the relations [4, 5]

$$\alpha_{pl} = \sum \tau_p b_l f(\boldsymbol{\tau}, \mathbf{b}; \mathbf{r}, t); \tag{1.6}$$

$$J_{kl} = \varepsilon_{pmk} \sum \tau_p b_l v_m(\boldsymbol{\tau}, \mathbf{b}) f(\boldsymbol{\tau}, \mathbf{b}; \mathbf{r}, t), \tag{1.7}$$

where τ is the unit vector of the tangent to the dislocation line; b is the Burgers vector; $v(\tau, b)$ is the vector of dislocation velocity; r is the radius vector of a point of the body; t is time. Summation is carried out in (1.6), (1.7) over all possible values of the vector pairs (τ, b). Equations (1.4)-(1.7) express the relationship between the macroscopic $\beta^P_{k\ell}$ and microscopic $f(\tau, b; r, t)$ characteristics.

The dislocation velocity vector $v(\tau, b)$ depends in the general case on the type of dislocation, i.e., on the vectors (τ, b). Let us examine the special case of a one-velocity flow of dislocations, when the velocity vector $v(\tau, b)$ is independent of (τ, b) for dislocations which are all identical. Then v_m is taken from under the summation sign in (1.7), and the remaining sum turns out to be equal to $\alpha_{p\ell}$. It follows from this that the tensors $\alpha_{p\ell}$ and $J_{k\ell}$ are related in this case:

$$J_{kl} = \varepsilon_{pmk}\alpha_{pl} v_m. \tag{1.8}$$

Inserting Eqs. (1.4) and (1.5) for the tensors $\alpha_{p\ell}$ and $J_{k\ell}$ into (1.8), after some simple transformations we obtain

$$\dot{\beta}^P_{kl} = -\beta^P_{kl,m} v_m + \beta^P_{ml,k} v_m. \tag{1.9}$$

Equations (1.9) can be regarded as the governing relation (in differential form) for plastic distortion $\beta^P_{k\ell}$, linking the tensor of the rate of plastic distortion $\dot{\beta}^P_{k\ell}$ with the tensor of the effective stresses σ^+_{ij}. The latter enter into (1.9) through the dependence of the dislocation velocity vector v on σ^+_{ij}. We will assume that the dynamic law for the dislocations is known either from experiment or from the microscopic theory of dislocation mobility (see [6], for example). Without concretizing until the end, we write the dynamic law for dislocations in the form

$$\mathbf{v} = A\left[\sigma^+_{ij}(\mathbf{r}, t)\right], \tag{1.10}$$

where A is a vector function of the effective plastic stress σ^+_{ij}. The effective stresses are made up of the external σ^0_{ij} and internal (connected with the incompatibility of the plastic strain) stresses σ_{ij}:

$$\sigma^+_{ij} = \sigma^0_{ij} + \sigma_{ij}. \tag{1.11}$$

As the external stresses, the internal stresses can be determined by means of (1.3). Here, instead of $\beta^0_{k\ell}$ we use the elastic distortions β_{mn} due to the incompatibility of the plastic distortion. According to [7], for β_{mn} we find that

$$\beta_{mn}(\mathbf{r}, t) = -\int c_{ijkl} G_{jn,im}(\mathbf{R}, T)\, \beta^P_{kl}(\mathbf{r}', t')\, d\mathbf{r}' dt' - \beta^P_{kl}(\mathbf{r}, t). \tag{1.12}$$

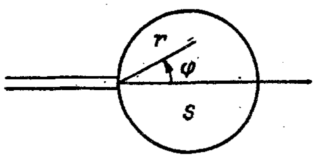

Fig. 1

Here, G_{jn} is the dynamic Green function of the elastic problem; $R = r - r'$; $T = t - t'$.

Thus, Eqs. (1.9), augmented by the dynamic law for the dislocations (1.10) and relations (1.1)-(1.3), (1.11), (1.12) for determination of the effective stresses, constitute a closed system which describes the development of plastic distortion over time. One feature of the system is that it describes plastic behavior directly on the basis of physical laws of dislocation motion — the dislocations being elementary carriers of plastic strain.

The unusual form of the equations obtained may cause some uncertainty. In fact, on the one hand, they include the dynamic law of dislocation motion (1.10). On the other hand, they do not include any information on dislocation multiplication or other changes in dislocations which occur in the general case [6]. The fact is that the above assumption regarding a one-velocity flow of dislocations precludes the multiplication of dislocations in a volume. Any physical process of multiplication in the volume of the body would lead to the formation of dislocations of a different sign. These dislocations, being located in the same field of effective stresses σ^{+}_{ij}, would also have velocity vectors of opposite sign. The latter result would contradict the assumption of a one-velocity dislocation flow. It is the exclusion of dislocation multiplication in the volume of the body (but not on its surface) which so significantly simplifies the description of plastic behavior. Here, dislocations are created on the surface of the body and propagate with the plastic distortion front. The gradient of plastic distortion completely determines the dislocation density, in accordance with (1.4). Dislocations of this type are referred to as geometrically necessary dislocations [8].

To avoid misunderstandings, we should discuss certain other important aspects of the problem. First, the assumption of a one-velocity flow actually pertains to the geometrically necessary dislocations. Geometrically necessary dislocations of one type can actually be represented by physical dislocations of several types. For example, a dislocation subboundary in a crystal with a given orientation can be represented by different systems of dislocations [6]. Second, although multiplication is prohibited for geometrically necessary dislocations in the volume of the body, processes involved in recharging are permitted for the physical dislocations which represent them. In these processes, dislocations are annihilated and generated as they interact. Such a situation may occur, for example, in a polycrystalline material in the successive movement of slip from grain to grain across the boundary between the grains.

On the whole, the equations obtained are applicable under conditions of large gradients of plastic distortion. Such conditions are realized, in particular, in the propagation of elastoplastic waves.

2. Now let us examine specific applications of the formalism presented above. As the first problem we take a shear crack in an elastoplastic body (Fig. 1). We will assume that the crack is semi-infinite and that the plastic zone S is small. Thus, the stresses in the plastic zone are completely determined by the stress intensity factor at the crack tip K [9]. We will represent the development of plastic flow as the radial motion of screw dislocations emitted by the crack tip. The dislocation density tensor $\alpha_{p\ell}$ has a unique nontrivial component $\alpha_{zz} = \alpha$, and we write Eq. (1.9) as follows (in cylindrical coordinates r, ϕ, z, with the z axis being normal to the plane of the drawing and directed along the crack front)

$$\dot{\beta}^{p}_{\varphi z} = -\beta^{p}_{\varphi z,r} v_r. \qquad (2.1)$$

Here, it is considered that the vector v has only a radial component v which depends on the shear stresses $\sigma_{\phi z} = \sigma$ in the radial plane. Let this dependence have the form

$$v_r = 0,\ \sigma^+ \leqslant \sigma_1;\ v_r = F(\sigma^+),\ \sigma^+ > \sigma_1,$$

where σ_1 is the critical stress associated with dislocation movement; $F(\sigma^+)$ is a scalar function. In the quasistatic case, by using Eqs. (1.1)-(1.3) and (1.12) we can obtain the following expressions for the external $\sigma^0_{z\phi}$ and internal $\sigma_{z\phi}$ stresses in the plastic zone

$$\sigma^0_{z\varphi} = -\operatorname{Im}\left\{\frac{1}{i}\frac{K}{\sqrt{2\pi\zeta}}e^{i\varphi}\right\},$$

$$\sigma_{z\varphi} = \int\limits_S M(r, \varphi; r', \varphi')\beta^P_{\varphi z, r}(r', \varphi')\,dS'.$$

Here, m is a kernel:

$$M(r, \varphi; r', \varphi') = -\operatorname{Im}\left\{\frac{1}{i}\frac{\mu}{2\pi}\sqrt{\frac{\zeta'}{\zeta}}\frac{1}{\zeta-\zeta'}e^{i\varphi}\right\};$$

$\zeta = re^{i\phi}$; $\zeta' = r'e^{i\phi'}$; μ is the shear modulus; S is the region occupied by the plastic zone in the plane r, ϕ; Im is the imaginary part of the complex function.

We will limit ourselves to the static solution corresponding to the state at $t \to \infty$, when the plastic zone is formed and dislocation motion has ceased. At $v_r = 0$ and $\dot{\beta}^P_{\phi z} = 0$, Eq. (2.1) is satisfied exactly, and the condition of limit equilibrium of the dislocations ($\sigma^+ = \sigma_1$) reduces to

$$\int\limits_S M(r, \varphi; r', \varphi')\beta^P_{\varphi z, r}(r', \varphi')\,dS' - \operatorname{Im}\left\{\frac{1}{i}\frac{K}{\sqrt{2\pi\zeta}}e^{i\varphi}\right\} = \sigma_1. \qquad (2.2)$$

As can be verified by direct substitution, the solution of (2.2) will be

$$\beta^P_{\varphi z} = \frac{K^2}{\pi\mu\sigma_1}\frac{\cos\varphi}{r},\quad r \leqslant R_0\cos\varphi,$$

where $R_0 = K^2/\pi\sigma_1^2$. This result agrees with the well-known solution obtained by the usual methods in [10].

As the second application, we will examine the propagation of a plane plastic wave in a uniformly stressed infinite body (Fig. 2). In contrast to the case of an elastic wave, propagation of a plastic wave is associated with energy dissipation, and nondecaying motion must be maintained as a result of the work of external forces (in the present case, uniform elastic tensile stresses σ_{yy} = const). Equation (1.9) allows the homogeneous solution $\beta^P_{k\ell}$ = const in the region where v = 0. The plastic wave can be represented as a piecewise-uniform solution. Let the plastic wave propagate along the x axis, and at the given moment of time let the wave front coincide with the plane x = 0 (see Fig. 2). For a wave of the simplest form, we put $\beta^P_{k\ell} = 0$ at x > 0 and $\beta^P_{yy} = \beta^P_-$ = const at x < 0. The remaining components of $\beta^P_{k\ell}$ are nontrivial. Inserting this discontinuous solution of (1.9) into (1.4), we find that the surface density of geometrically necessary dislocations on the front of the plastic wave

$$\alpha_{zy} = \beta^P_-\delta(x), \qquad (2.3)$$

where $\delta(x)$ is the Dirac delta function. A dislocation density of the form (2.3) is also characteristic of the dislocation model of a shock front proposed by Smith [1].

To determine the rate of propagation w of the front of the plastic wave, we need to assign a physical set of dislocations which has the necessary density (2.3) and can move conservatively in the ositive direction of the x axis under the influence of uniform tensile stresses σ^0_{yy} = const. This requirement is satisfied by a system of two kinds of edge dislocations sliding in planes inclined at a 45° angle to the y axis (see Fig. 2). Due to symmetry, each of these dislocation systems has the same distribution function $f(\tau, b; r, t) = \beta^P_-/2b_y\ \delta(x - wt)$ and velocity component v_x. Equating the dislocation flux densities (1.7),

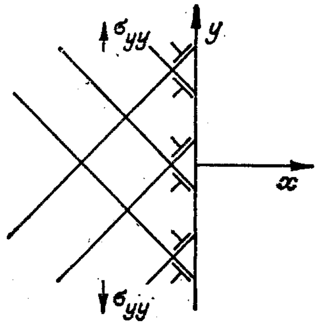

Fig. 2

(1.8), we find $w = v_x = v/\sqrt{2}$ (v is the velocity of the physical dislocation in the stress field σ^0_{yy} = const).

In reality, plastic waves can also occur in the propagation of an elastis shock wave [11]. In connection with this, it is interesting to examine the rate of erosion of the plastic wave front. To do this, we assume $\beta^P_{yy} = \beta(x, t)$ to now be a smooth function. The other assumptions made previously remain in force. The below relative to $\beta(x, t)$ then follows from (1.9)

$$\dot{\beta}^P + v_x \beta^P_{,x} = 0. \tag{2.4}$$

Here, $v_s = v_x(x, t)$ depends on x and t. The function $\beta^P(x, t)$ satisfies the boundary conditions

$$\beta^P(-\infty, t) = \beta^P_-, \quad \beta^P(\infty, t) = 0.$$

We take the following power law for the dislocation velocity [6]

$$v = v_0 \left(\frac{\sigma^+}{\sigma_0}\right)^m, \tag{2.5}$$

where v_0, σ_0, and m are constants; σ^+ are the effective shear stresses ($\sigma^+ = 1/\sqrt{2}\ \sigma^+_{yy}$). We have the following for σ^+_{yy} in the isotropic case

$$\sigma^+_{yy} = \sigma^0_{yy} - \frac{\mu}{1-\nu}\beta^P \tag{2.6}$$

(ν is the Poisson's ratio).

Let $x = x(\beta^P_0, t)$ describe the motion of a point of constant value $\beta^P = \beta^P_0$. Then, with allowance for (2.5) and (2.6), the below equation follows from (2.4) [12]

$$\frac{dx}{dt} = \frac{v_0}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}\,\sigma_0}\left[\sigma^0_{yy} - \frac{\mu}{1-\nu}\beta^P(x_0)\right]\right\}^m, \tag{2.7}$$

$\beta^P(x_0) = \beta^P(x = x_0, t = 0)$. We write the solution of (2.7) as

$$x = x_0 + \frac{v_0}{\sqrt{2}}\left\{\frac{1}{\sqrt{2}\,\sigma_0}\left[\sigma_{yy} - \frac{\mu}{1-\nu}\beta^P(x_0)\right]\right\}^m t.$$

We then put m = 1. Let

$$\beta^P(x_0) = \beta^P_-, \quad x_0 < 0;$$
$$\beta^P(x_0) = \beta^P_- - \lambda x_0, \quad 0 \leqslant x_0 \leqslant \Delta x_0; \quad \beta^P(x_0) = 0, \quad x_0 > \Delta x_0,$$

where $\Delta x_0 = \beta^P_-/\lambda$ is the initial width of the plastic wave front; λ is a constant. In this case, we find the following from (2.7) for the width of the plastic wave front at an aritrary moment of time t

$$\Delta x(t) = \Delta x_0 + \frac{v_0}{2} \frac{\mu}{(1-\nu)\sigma_0} \beta^P_- t.$$

As is evident, $\Delta x(t)$ increases in proportion to the time t and the amplitude of the plastic wave β^P_-.

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NUMERICAL ANALYSIS OF THE NONLINEAR STABILITY OF VIBRATIONS IN A PLATE LYING ON A LAYER OF VISCOUS, COMPRESSIBLE LIQUID

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Problems related to the stability of vibrations of mechanical systems that are in contact with a viscous, compressible liquid often arise in many areas of science, engineering, and contemporary production. A typical example is the problem concerning the stability of heavily loaded friction nodes under conditions of increased vibration.

In order to take into account the compressibility of liquid described by the Newtonian model with linear viscosity, one must consider both the shearing viscous stresses and the volumetric viscous stresses (which is usually not the case) [1]. The assumption that the coefficient of volumetric viscosity is zero is in most cases unjustified, and for some liquids the coefficient of volumetric viscosity can be many times (sometimes many orders) greater than the coefficient of ordinary shearing viscosity. Also, when the forces that act on a liquid are intense, one cannot ignore the dissipation of energy for a change in volume. For vibrational processes that are accompanied by a change in volume, the effect of volumetric viscosity can be very substantial.

1. Formulation of the Problem and a Determination of Equations. We will consider the one-dimensional problem of forced vibrations in a massive layer S that lies on a layer of viscous compressible liquid (Fig. 1) acted on by a periodic force F(t).

The basic equations of the problem are:

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